RAMAKRISHNA MISSION VIDYAMANDIRA

(A Residential Autonomous College under University of Calcutta)

First Year

First-Semester Examination, December 2010

Date : 15-12-2010 **MATHEMATICS (Honours)** Full Marks : 100

Time : 11am – 3pm **Paper - I**

(Use separate answer script for each group)

Group – A

1. a) Answer **any one** question :

   i) Let S be a non-empty subset of $\mathbb{R}$, ($\mathbb{R}$ is the set of real numbers) bounded above and $T = \{-x : x \in S\}$. Prove that the set T is bounded below and inf T = – sup S. [2]

   ii) Prove that for a positive real number x there exists a natural number m such that $m-1 \le x < m$. [2]

   b) Answer **any one** question :

   i) Prove that the set of rational numbers is enumerable. [3]

   ii) Prove that $\lim\limits_{x\to 0} \sin\frac{1}{x}$ does not exist. [3]

Answer **any three** from Question no. 2 to Question no. 6 [3×5 = 15]

2. a) Show that the union of finite number of open sets of real numbers is an open set.

   b) If A and B are subsets of real numbers prove that $Cl(A \cap B) \subseteq Cl(A) \cap Cl(B)$. [3+2]

3. State and prove Bolzano-Weierstrass theorem for infinite point sets. [1+4]

4. a) Show that a monotone increasing sequence of real numbers which is bounded above is convergent.

   b) Prove that the sequence $\{u_n\}_{n\in\mathbb{N}}$ defined by $u_1 = \sqrt{6}$, $u_{n+1} = \sqrt{6+u_n}$ for $n \ge 1$ is convergent.[3+2]

5. a) Show that a bounded sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent if and only if $\overline{\lim}\, x_n = \underline{\lim}\, x_n$.

   b) Let I = (a,b) be a bounded open interval and $f : I \to \mathbb{R}$ be a monotone increasing function on I. If f is bounded below on I, then show that $\lim\limits_{x\to a+} f(x) = \inf\limits_{x\in(a,b)} f(x)$. [3+2]

6. a) Let $S = \left\{ \frac{1}{m} + \frac{1}{n} ; m, n \text{ are narural numbers} \right\}$. Show that 0 is a limit point of S.

   b) Using Cauchy criterion for the existence of limit, prove that for the function $f : (0,1) \to \mathbb{R}$ defined by, f(x) = 1, if x is rational, $x \in (0,1)$.

   = –1, if x is irrational, $x \in (0,1)$

   $\lim\limits_{x\to a} f(x)$ does not exist when $a \in [0,1]$. [2+3]

7. Answer **any two** questions : [2×3 = 6]

   a) For any three sets A, B, C prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$. [3]

   b) Prove that each partition of a set S yields an equivalence relation on S. [3]

c) Let A, B be two non-empty sets and $f : A \to B$ be a bijective mapping. Prove that the mapping $f^{-1} : B \to A$ is also a bijection. [3]

Answer **any four** questions from Question no. 8 to Question no. 14. [6×4 = 24]

8. a) Let X be a non-empty set. Prove that there does not exist any surjective map from X onto P(X), the power set of X. [3]

   b) Prove that the relation ρ on $\mathbb{R}$ defined by '$x\rho y$ iff $x - y \in Q\ (x, y \in \mathbb{R})$' is an equivalence relation. Find the equivalence class containing the element '0'. [2+1]

9. a) Prove that the necessary and sufficient condition, that a non-empty subset H of a group (G, ·) is a subgroup of G is that $a, b \in H \Rightarrow a.b^{-1} \in H$.

   b) Let $a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 4 & 7 & 5 & 2 & 3 & 1 \end{pmatrix}$ be an element of S_7. Is a^{-1} an even permutation? [4+2]

10. Define the centre of a group. If G is a non-commulative group of order 10 then show that G has a trivial centre. [1+5]

11. a) In a group (G, ·) $a^3 = e$ (the identity of G) and $a.b.a^{-1} = b^2$ $(a, b \in G)$; find the order of b if $b \neq e$.

    b) If (H, ·) be a sub-group of a group (G, ·), prove that the two left-cosets of H in G are either disjoint or identical. [2+4]

12. Show that all the proper subgroups of the symmetric group S_3 are cyclic. Prove that every cyclic group is abelian. Give an example of an abelian group which is not cyclic. [2+2+2]

13. a) Let n > 1 be an integer. Prove that the characteristic of the ring Z_n is n.

    b) Prove that a finite integral domain is a field. [2+4]

14. a) Give examples of two subfields K_1, K_2 of a field F such that $K_1 \cup K_2$ fails to be a subfield of F.

    b) Prove that the field Q of all rational numbers has no proper subfield.

    c) Give an example with justification of a finite field containing more than 25 elements. [2+2+2]

<u>Group – B</u>

Answer Question No. 15 and 20 and <u>any two</u> from Question No. 16-19

15. Answer **<u>any one</u>** question :

    a) Obtain the differential equation of all circles each of which touches the axis of x at the origin. [5]

    b) Find an integrating factor of the differential equation :
    $(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0$ and hence solve it. [2+3 = 5]

16. a) Reducing the differential equation $x^2p^2 + py(2x+y) + y^2 = 0$ to Clairaut's form by the Substitutions $y = u$, $xy = v$, solve it and prove that $y+4x = 0$ is a singular solution. [3+2]

    b) By the use of symbolic operator D, find the solution of the equation :
    $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$. [5]

17. a) Show that if y_1 and y_2 be solutions of the equation $\frac{dy}{dx}+Py=Q$ where P and Q are functions of x alone and $y_2 = y_1Z$ then $Z=1+ae^{-\int\frac{Q}{y_1}dx}$ (a is an arbitrary constant). [5]

b) By the method of undetermined coefficients find the solution of the equation :
$\frac{d^3y}{dx^3}+2\frac{d^2y}{dx^2}-\frac{dy}{dx}-2y=e^x+x^2.$ [5]

18. a) Solve by the method of variation of parameters, $\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}y=\log x\,(x>0)$ [5]

b) Show that the equation $(1+x+x^2)\frac{d^3y}{dx^3}+(3+6x)\frac{d^2y}{dx^2}+6\frac{dy}{dx}=0$ is exact and hence solve it. [1+4]

19. a) Find an integrating factor of the equation $(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$ and hence solve it. [2+3]

b) Solve : $y\frac{d^2y}{dx^2}-\left(\frac{dy}{dx}\right)^2=y^2\log x$. [5]

20. Answer **any five** questions :

a) Line segments are drawn from a vertex of a parallelogram to the mid-points of the opposite sides. Show that they trisect a diagonal. [5]

b) If three vectors $\vec{a}, \vec{b}, \vec{c}$ be expressed as a linear combination of three non-coplanar vectors $\vec{l}, \vec{m}, \vec{n}$ as $\vec{a}=a_1\vec{l}+a_2\vec{m}+a_3\vec{n}$, $\vec{b}=b_1\vec{l}+b_2\vec{m}+b_3\vec{n}$ and $\vec{c}=c_1\vec{l}+c_2\vec{m}+c_3\vec{n}$ then show that their scalar product $[\vec{a}\,\vec{b}\,\vec{c}]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}[\vec{l}\,\vec{m}\,\vec{n}]$ [5]

c) i) Prove that $(\vec{a}\times\vec{b})\times\vec{c}=\vec{a}\times(\vec{b}\times\vec{c})$ if and only if $(\vec{c}\times\vec{a})\times\vec{b}=\vec{0}$ [3]

ii) Solve for the scalars y and z satisfying $y\vec{a}+z\vec{b}=\vec{c}$, where $\vec{a},\vec{b}$ are given non-collinear vectors. [2]

d) A force of magnitude 15 unit acts in the direction of the vector $\hat{\imath}-2\hat{\jmath}+2\hat{k}$ and passes through the point $2(\hat{\imath}-\hat{\jmath}+\hat{k})$. Find the moment of the force about the point $\hat{\imath}+\hat{\jmath}+\hat{k}$. [5]

e) Show that the necessary and sufficient condition that a proper vector $\vec{u}$ always remains parallel to a fixed line is that $\vec{u}\times\frac{d\vec{u}}{dt}=\vec{0}$. [5]

f) i) Find the maximum value of the directional derivative of $\phi=x^2+z^2-y^2$ at the point (1,3,2). Find also the direction in which it occurs. [1+1]

ii) Prove that $\text{Curl}\left\{\frac{\vec{a}\times\vec{r}}{r^3}\right\}=\frac{\vec{a}}{r^3}+\frac{3}{r^5}(\vec{a}.\vec{r})\vec{r}$ where $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ and $\vec{a}$ is a constant vector. [3]

g) Find the shortest distance between the two lines through A(6, 2, 2) and C(–4, 0, –1) and parallel to the vectors (1, –2, 2) and (3, –2, –2) respectively. Find the points where the lines meet the line of shortest distance. [5]

h) Show that the vector field determined by grad f is always irrotational and the vector field determined by curl $\vec{F}$ is always solenoidal. [5]